Generalized Neutrosophic Set and Generalized Neutrosophic Topological Spaces

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Abstract In this paper we introduce definitions of generalized neutrosophic sets. After given the fundamental definitions of generalized neutrosophic set operations, we obtain several properties, and discussed the relationship between generalized neutrosophic sets and others. Finally, we extend the concepts of neutrosophic topological space [9], intuitionistic fuzzy topological space [5, 6], and fuzzy topological space [4] to the case of generalized neutrosophic sets. Possible application to GIS topology rules are touched upon.

Keywords Neutrosophic Set, Generalized Neutrosophic Set, Neutrosophic Topology

1. Introduction

Neutrosophy has laid the foundation for a whole family of new mathematical theories generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory. The fuzzy set was introduced by Zadeh [10] in 1965, where each element had a degree of membership. The intuitionistic fuzzy set (IFS for short) on a universe X was introduced by K. Atanassov [1, 2, 3] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. After the introduction of the neutrosophic set concept [7, 8, 9]. In this paper we introduce definitions of generalized neutrosophic sets. After given the fundamental definitions of generalized neutrosophic set operations, we obtain several properties, and discussed the relationship between generalized neutrosophic sets and others. Finally, we extend the concepts of neutrosophic topological space [9].

2. Terminologies

We recollect some relevant basic preliminaries, and in particular, the work of Smarandache in [7, 8], Atanassov in [1, 2, 3] and Salama [9]. Smarandache introduced the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where \( 0^*,1^* \) is nonstandard unit interval.

Definition [7, 8]

Let T, I, F be real standard or nonstandard subsets of \( 0^*,1^* \) with
\[
\begin{align*}
\text{Sup}_T &= t_{\text{sup}}, \text{Inf}_T = t_{\text{inf}} \\
\text{Sup}_I &= i_{\text{sup}}, \text{Inf}_I = i_{\text{inf}} \\
\text{Sup}_F &= f_{\text{sup}}, \text{Inf}_F = f_{\text{inf}} \\
n_{\text{sup}} &= t_{\text{sup}} + i_{\text{sup}} + f_{\text{sup}} \\
n_{\text{inf}} &= t_{\text{inf}} + i_{\text{inf}} + f_{\text{inf}} \\
T, I, F &\text{ are called neutrosophic components}
\end{align*}
\]

Definition [9]

Let X be a non-empty fixed set. \( A \) neutrosophic set (NS for short) \( A \) is an object having the form
\[
A = \left\{ (x, \mu_\delta(x), \sigma_\delta(x), \gamma_\delta(x)) : x \in X \right\}
\]

Where \( \mu_\delta(x) \), \( \sigma_\delta(x) \) and \( \gamma_\delta(x) \) which represent the degree of membership function (namely \( \mu_A(x) \)), the degree of indeterminacy (namely \( \sigma_A(x) \)), and the degree of non-membership (namely \( \gamma_A(x) \)) respectively of each element \( x \in X \) to the set \( A \).

Definition [9]

The NSS \( 0_N \) and \( 1_N \) in \( X \) as follows:
\[
\begin{align*}
0_N &\text{ may be defined as:} \\
0_{(0)} &= \left\{ (x, 0, 0, 1) : x \in X \right\} \\
0_{(1)} &= \left\{ (x, 0, 1, 1) : x \in X \right\} \\
0_{(3)} &= \left\{ (x, 0, 1, 0) : x \in X \right\} \\
0_{(4)} &= \left\{ (x, 0, 0, 0) : x \in X \right\} \\
1_N &\text{ may be defined as:} \\
1_{(0)} &= \left\{ (x, 1, 0, 0) : x \in X \right\}
\end{align*}
\]
(1.) \( I_N = \{ (x, 1, 0, 1) : x \in X \} \)

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3. Generalized Neutrosophic Sets

We shall now consider some possible definitions for basic concepts of the generalized neutrosophic set.

**Definition**

Let \( X \) be a non-empty fixed set. A generalized neutrosophic set (GNS for short) \( A \) is an object having the form \( A = \{ (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X \} \). Where \( \mu_A(x) \), \( \sigma_A(x) \) and \( \gamma_A(x) \) represent the degree of membership function (namely \( \mu_A(x) \)), the degree of indeterminacy (namely \( \sigma_A(x) \)), and the degree of non-membership (namely \( \gamma_A(x) \)) respectively of each element \( x \in X \) to the set \( A \) where the functions satisfy the condition \( \mu_A(x) \land \sigma_A(x) \land \nu_A(x) \leq 0.5 \).

**Remark**

A generalized neutrosophic \( A = \{ (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X \} \) can be identified to an ordered triple \( < \mu_A, \sigma_A, \gamma_A > \) in \( \mathbb{I} = [0, 1] \) on \( X \), where the triple functions satisfy the condition \( \mu_A(x) \land \sigma_A(x) \land \nu_A(x) \leq 0.5 \).

**Remark**

For the sake of simplicity, we shall use the symbol \( A <= x, \mu_A(x), \sigma_A(x), \gamma_A(x) > \) for the GNS \( A = \{< x, \mu_A(x), \sigma_A(x), \gamma_A(x) > : x \in X \} \).

**Example**

Every GIFS \( A \) a non-empty set \( X \) is obviously on GNS having the form

\( A = \{< x, \mu_A(x), 1-\mu_A(x), \gamma_A(x), \nu_A(x) > : x \in X \} \)

**Definition**

Let \( A = \{ \mu_A, \sigma_A, \gamma_A \} \) a GNS on \( X \), then the complement of the set \( A \) (\( C(A) \), for short) may be defined as three kinds of complements:

\( C_1(A) = \{ (x, 1- \mu_A(x), \sigma_A(x), 1- \nu_A(x)) : x \in X \} \)

\( C_2(A) = \{ (x, \nu_A(x), \sigma_A(x), \mu_A(x)) : x \in X \} \)

\( C_3(A) = \{ (x, \nu_A(x), 1- \sigma_A(x), \mu_A(x)) : x \in X \} \)

One can define several relations and operations between GNS as follows:

**Definition**

Let \( X \) be a non-empty set, and GNS \( A \) and \( B \) in the form \( A = \{ x, \mu_A(x), \sigma_A(x), \gamma_A(x) \} \),

\( B = \{ x, \mu_B(x), \sigma_B(x), \gamma_B(x) \} \), then we may consider two possible definitions for subsets \( A \subseteq B \)

\( A \subseteq B \) may be defined as

\( A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \gamma_A(x) \geq \gamma_B(x) \) and \( \sigma_A(x) \leq \sigma_B(x) \)

\( \forall x \in X \)

**Proposition**

For any generalized neutrosophic set \( A \) the following are holds

\( 0_N \subseteq A \). \( 0_N \subseteq 0_N \)

\( A \subseteq 1_N \). \( 1_N \subseteq 1_N \)

**Definition**

Let \( X \) be a non-empty set, and

\( A = < x, \mu_A(x), \gamma_A(x), \sigma_A(x) > \),

\( B = < x, \mu_B(x), \gamma_B(x), \sigma_B(x) > \) are GNS. Then \( A \cap B \) maybe defined as:

\( (I ) ) A \cap B = < x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \land \sigma_B(x) \>

\( (II ) ) A \cap B = < x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \land \sigma_B(x) \>

\( (III ) ) A \cap B = < x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \land \sigma_B(x) \>

\( A \cup B \) may be defined as:

\( (U_1) A \cup B = < x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \>

\( (U_2) A \cup B = < x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \>

\( A \cup B \) maybe defined as:

\( (U_3) A \cup B = < x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \lor \gamma_B(x), \sigma_A(x) \lor \sigma_B(x) \>

\( \mu_A(x) \lor \sigma_A(x) \lor \nu_A(x) \leq 0.5 \)

Then the family \( G = \{ O, A \} \) is a GNS on \( X \).

We can easily generalize the operations of generalized intersection and union in definition 3.4 to arbitrary family of GNS as follow:

**Definition**

Let \( \{ A_j : j \in J \} \) be an arbitrary family of GNS in \( X \), then

\( \bigcap_{j \in J} A_j \) maybe defined as:

\( 1) \bigcap_{j \in J} A_j \leftarrow x, \land \mu_{A_j}(x), \land \sigma_{A_j}(x), \lor \gamma_{A_j}(x) \)
2) \( \bigcap A_j = \left\{ x, \wedge \mu_{A_j}(x), \vee \sigma_{A_j}(x), \vee \gamma_{A_j}(x) \right\} \)
\( \bigcup A_j \) may be defined as:
1) \( \bigcup A_j = \left\{ x, \vee_{j \in J} \mu_{A_j}, \wedge \sigma_{A_j}, \wedge \gamma_{A_j} \right\} \)
2) \( \bigcup A_j = \left\{ x, \vee_{j \in J} \mu_{A_j}, \vee \sigma_{A_j}, \wedge \gamma_{A_j} \right\} \)

**Definition**

Let \( A \) and \( B \) are generalized neutrosophic sets then \( A \subseteq B \) may be defined as
\( A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\} \). Then the neutrosophic open set (NOS) is known as neutrosophic open set (NTS) of generalized neutrosophic sets, \( NCS \) and \( NOS \). It can be also shown that \( \subseteq \) is a family \( \bigcup_{i \in J} G_i \in \tau \) for all \( i \in J \).

**Proposition**

For all \( A, B \) two generalized neutrosophic sets then the following are true
i) \( C(A \cap B) = C(A) \cap C(B) \)
ii) \( C(A \cup B) = C(A) \cup C(B) \)

**4. Generalized Neutrosophic Topological Spaces**

Here we extend the concepts of and intuitionistic fuzzy topological space [5, 7], and neutrosophic topological space [9] to the case of generalized neutrosophic sets.

**Definition**

A generalized neutrosophic topology (GNT for short) an a non empty set \( X \) is a family \( \tau \) of generalized neutrosophic subsets in \( X \) satisfying the following axioms

\( (GNT_1) \) \( \emptyset, X \in \tau \),
\( (GNT_2) \) \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \),
\( (GNT_3) \) \( \bigcup G_i \in \tau \ \forall \{G_i : i \in J\} \subseteq \tau \)

In this case the pair \((X, \tau)\) is called a generalized neutrosophic topological space (GNTS for short) and any neutrosophic set in \( \tau \) is known as neutrosophic open set (NOS for short) in \( X \). The elements of \( \tau \) are called open generalized neutrosophic sets, A generalized neutrosophic set \( F \) is closed if and only if \( C(F) \) is generalized neutrosophic open.

**Remark** A generalized neutrosophic topological spaces are very natural generalizations of intuitionistic fuzzy topological spaces allow more general functions to be members of intuitionistic fuzzy topology.

**Example**

Let \( X = \{x\} \) and \( A = \{x, 0.5, 0.5, 0.4\} : x \in X \}
\( B = \{x, 0.4, 0.6, 0.8\} : x \in X \}
\( D = \{x, 0.5, 0.6, 0.4\} : x \in X \}
\( C = \{x, 0.4, 0.5, 0.8\} : x \in X \}

Then the family \( \tau = \{0, 1, A, B, C, D\} \) of GNTSs in \( X \) is generalized neutrosophic topology on \( X \).

**Example**

Let \( (X, \tau_0) \) be a fuzzy topological space in changes [4] sense such that \( \tau_0 \) is not indiscrete suppose now that \( \tau_0 = \{0, 1, A, N\} \cup \{V_j : j \in J\} \) then we can construct two GNTSs on \( X \) as follows
\( \tau_0 = \{0, 1, A, N\} \cup \{V_j : j \in J\} \)
\( \tau_0 = \{0, 1, A, N\} \cup \{V_j, 0, \sigma(x), 1 - V_j : j \in J\} \)

**Proposition**

Let \( (X, \tau) \) be a GNT on \( X \), then we can also construct several GNTSs on \( X \) in the following way:
a) \( \tau_{0,1} = \{\tau : \tau \subseteq \tau\} \),
b) \( \tau_{0,2} = \{\subseteq \tau : \tau \subseteq \tau\} \).

**Proof** a)
\( (GNT_1) \) and \( (GNT_2) \) are easy.
\( (GNT_3) \) Let \( \{G_j : j \in J, G_j \in \tau \} \subseteq \tau_{0,1} \). Since
\( \bigcup G_j \subseteq \tau_{0,1} \) we have
\( \bigcup G_j \subseteq \tau_{0,1} \)
This similar to (a)

**Definition**

Let \( \cap \tau_j \) be two generalized neutrosophic topological spaces on \( X \). Then \( \cap \tau_j \) is the coarsest NT on \( X \) containing all \( \tau_j \).

**Proof**. Obvious

**Definition**

The complement of \( A \) \( \{C(A) \} \) for short) of NOSs, \( A \) is called a generalized neutrosophic closed set (GCS for short) in \( X \).

Now, we define generalized neutrosophic closure and interior operations in generalized neutrosophic topological spaces:

**Definition**

Let \( (X, \tau) \) be GNTS and \( A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x)\} \) be a GNS in \( X \).

Then the generalized neutrosophic closh and generalized neutrosophic interior of \( A \) are defined by
\( G\text{NCl}(A) = \cap \{K : K \text{ is an NCS in } X \text{ and } A \subseteq K\} \)
\( G\text{NInt}(A) = \cup \{G : G \text{ is an NOS in } X \text{ and } G \subseteq A\} \) It can also be shown
that
\( G\text{NCl}(A) \) is an NCS and \( G\text{NInt}(A) \) is a GNTS in \( X \).

Let \( A \) be in \( X \) if and only if \( G\text{NCl}(A) \).
\( A \) is GNS in \( X \) if and only if \( G\text{NInt}(A) = A \).

**Proposition**
For any generalized neutrosophic set \( A \) in \((X, \tau)\) we have
(a) \( G\text{NCI}(C(A)) = C(G\text{NInt}(A)) \),
(b) \( G\text{NInt}(C(A)) = C(G\text{NCI}(A)) \).

**Proof.**
Let \( A = \{< x, \mu_A, \sigma_A, \nu_A >: x \in X \} \) and suppose that the family of generalized neutrosophic subsets contained in \( A \) are indexed by the family of G NSS contained in \( A \) are indexed by the family
\[ A = \{< x, \mu_{G_i}, \sigma_{G_i}, \nu_{G_i} >: i \in J \} \].
Then we see that
\[ G\text{NInt}(A) = \{< x, \vee \mu_{G_i}, \vee \sigma_{G_i}, \vee \nu_{G_i} >\} \] and hence
\[ C(G\text{NInt}(A)) = \{< x, \wedge \mu_{G_i}, \wedge \sigma_{G_i}, \wedge \nu_{G_i} >\} \].
Since \( C(A) \) and \( \mu_{G_i} \leq \mu_A \) and \( \nu_{G_i} \geq \nu_A \) for each \( i \in J \), we obtaining \( C(A) \), i.e.
\[ G\text{NCI}(C(A)) = \{< x, \wedge \nu_{G_i}, \vee \sigma_{G_i}, \wedge \mu_{G_i} >\} \].
Hence
\[ G\text{NCI}(C(A)) = C(G\text{NInt}(A)) \], follows immediately.
This is analogous to (a).

**Proposition**
Let \( (X, \tau) \) be a G NTS and \( A, B \) be two neutrosophic sets in \( X \). Then the following properties hold:
\( G\text{NInt}(A) \subseteq A \),
\( A \subseteq G\text{NCI}(A) \),
\( A \subseteq B \Rightarrow G\text{NInt}(A) \subseteq G\text{NInt}(B) \),
\( A \subseteq B \Rightarrow G\text{NCI}(A) \subseteq G\text{NCI}(B) \),
\( G\text{NInt}(G\text{NInt}(A)) = G\text{NInt}(A) \wedge G\text{NInt}(B) \),
\( G\text{NCI}(A \cup B) = G\text{NCI}(A) \vee G\text{NCI}(B) \),
\( G\text{NInt}(1_N) = 1_N \),
\( G\text{NCI}(O_N) = O_N \).

**Proof** (a), (b) and (e) are obvious (c) follows from (a) and (d).

**REFERENCES**


