THE PROBLEM OF THE QUEUEING SYSTEM $E_k/M/1$ WITH FINITE WAITING ROOM AND ADDITIONAL SERVERS FOR A LONGER QUEUE

G.S. MOKADDIS
Ain Shams University,
Department of Mathematics,
Faculty of Science,
Cairo,
Egypt

C.H. MATTAR
Cairo University,
Department of Statistics,
Faculty of Economics
and Political Sciences,
Cairo,
Egypt

M.M. EL-GENAIDY
Suez Canal University,
Department of Mathematics,
Faculty of Education,
Port-Said,
Egypt

ABSTRACT - This paper deals with an $E_k/M/1$ queueing system with additional servers for a longer queue with finite waiting room. The expected number of customers in the system in the long run, the expected waiting time of any customer until entering service, and the probability that a particular server is busy at any time are obtained under the assumption that the number of additional servers is dependent upon the number of customers in the system. The conditions under which the $E_k/M/1$ queueing system with additional servers is preferred to the $E_k/M/1$ queueing system with no additional server are discussed when both systems have the same finite waiting room, interarrival rate and different traffic intensities. There are two different methods to solve this problem.
FORMULATION OF THE PROBLEM

In many situations the number of servers is dependent on the queue length. One of those situations is dealt with in this paper, considering the traffic intensity of the system dependent on the number of additional servers. Suppose that the arrival occurs once at a time in Erlangian distribution. These customers get served at a single contour. Their service times are independent and identically exponentially distributed. The first to come is the first to be served. As long as there are enough customers for service, the service completion occur in a Poisson process. There is only one server as long as the number of customers in the system is greater than or equal to zero and less than or equal to N. As the number of customers in the system increases to more than N and is still less than or equal to 2N, an additional server is added. This additional server is removed as soon as the number of customers in the system decreases to N or less. Similarly if the number of customers in the system increases to more than 2N but is still less than or equal to 3N, the number of servers will be three. Generally speaking, as long as the number of customers in the system is less than or equal to (j+1)N but greater than jN, there are (j+1) servers in the system, where j=0, 1, 2, ..., (i-1). Suppose that the waiting room is of size K (K is a positive integer), and that a customer arriving to find the waiting room full departs immediately without waiting for service. There may be at most K+1 customers present at any time; K waiting and one in service. Therefore the number (j+1)N is must be equal or less than the number K+1 for all values of j. There are i servers in the system whenever the number of customers is greater than (i-1)N. Let $Q^*$ and Q be the number of customers in the long run in the $E_0/M/1$ queueing system with additional servers and the $E_0/M/1$ queueing system with no additional servers respectively, given that both systems have the same finite waiting room K, the same interarrival rate $\lambda$, and different traffic intensities.
PREFERENCE BETWEEN THE $E_k/M/1$ QUEUE WITH ADDITIONAL SERVERS FOR A LONGER QUEUE WITH FINITE WAITING ROOM AND THE CLASSICAL $E_k/M/1$ QUEUE

1. THE FIRST METHOD

The interarrival time obeys the Erlangian distribution whose probability density function, $a(t)$ is given by:

$$a(t) = \frac{k \lambda_n (k \lambda_n t)^{k-1} e^{-k \lambda_n t}}{(k-1)!} \quad \text{for} \quad t \geq 0.$$

But, the service time obeys the exponential distribution whose probability density function, $b(x)$ is given by:

$$b(x) = \mu_n e^{-\mu_n x},$$

where

$$\lambda_n = \lambda \quad \text{for every} \quad n \geq 0,$$

and

$$\mu_n = (j+1)\mu \quad \text{for} \quad jN < n \leq (j+1)N \ ; \ j = 0, 1, \ldots, (i-1).$$

The arriving facility is an $k$ stages Erlangian facility. An arriving customer is immediately inserted from the left side of the arriving facility and pass through exponential stages with parameter $\lambda$. As long as the number of customers in the system is less than or equal to $(j+1)N$ but greater than $jN$, there are $(j+1)$ servers in the system. There are $i$ servers in the system whenever the number
of customers is greater than \((i-1)N\). The waiting room is finite and the traffic intensity is dependent on the number of available additional servers. On exiting from the right side of arriving facility, an arrival to the given system \(E_{0}/M/1\) is said to occur. No other customer can be inserted from the left side into the arriving facility before the previous one exits from the right side of this facility. Once having arrived, the customer joining the queue, waits for service and is then served according to the distribution \(b(x)\).

The steady state probabilities \(P_m = \lim_{t \to \infty} P_m(t)\), here \(P_m(t) = \Pr(\text{m customers in the system at time } t)\), are given by Leonard \([1]\) as:

\[
P_m = \sum_{n=m}^{\infty} \lambda^n / n!,
\]

where \(P_n = \lim_{t \to \infty} P_n(t)\) and \(P_n(t) = \Pr(\text{n arrival stages in the system at time } t)\).

The probabilities \(P_n\) are derived from their Z-transform \(P_n(z); n = 0, 1, 2, \ldots\).

The equilibrium equations can be written as:

\[
k \lambda \lambda P_n = \mu P_{n-1}, \quad n = 0
\]

\[
k \lambda \lambda P_n = k \lambda P_{n-1} + \mu P_{n+k}, \quad 1 \leq n \leq k - 1
\]

\[
(k \lambda + \mu) P_n = k \lambda P_{n-1} + \mu P_{n+k}, \quad k - 1 < n \leq N
\]

\[
(k \lambda + 2 \mu) P_n = k \lambda P_{n-1} + 2 \mu P_{n+k}, \quad N < n \leq 2N
\]

\[
[k \lambda + (i+1) \mu] P_n = k \lambda P_{n-1} + (i+1) \mu P_{n+k}, \quad jN < n \leq (j+1)N
\]

\[
[k \lambda + (i-1) \mu] P_n = k \lambda P_{n-1} + (i-1) \mu P_{n+k}, \quad (i-2)N < n \leq (i-1)N
\]
\[(k \lambda + i \mu) P_n = k \lambda P_{n-1} + i \mu P_{n+K}, \quad (i-1)N < n \leq K. \quad (1.6)\]

Operating upon the equations in (1.2), adding and subtracting the missing terms as appropriate, yields:

\[
\sum_{n=1}^{N} ((\mu + k \lambda) P_n Z^n - \sum_{n=1}^{k-1} \mu P_n Z^n) = \sum_{n=1}^{N} k \lambda P_{n-1} Z^n + \sum_{n=1}^{N} \mu P_{n+K} Z^n.
\]

Therefore,

\[
(\mu + k \lambda)(P(z) - P_0) - \sum_{n=1}^{k-1} \mu P_n Z^n = k \lambda Z P(z) + \frac{\mu}{Z^k} \left[ P(z) - \sum_{n=0}^{k} P_n Z^n \right].
\]

Thus,

\[
P(z) = \frac{(1-z^k) \sum_{n=0}^{k-1} P_n Z^n}{k \rho_0 Z^{k+1} - (1 + \rho_0 k) Z^k + 1},
\]

where \( \rho_0 = \lambda / \mu \).

Then \( P(z) \), by Leonard [1], can be written as:

\[
P(z) = \frac{(1-Z^k)}{C_0(1-Z)(1-Z/Z_0)},
\]

where \( C_0 \) is a constant and \( Z_0 \) is a solution for the equation:

\[k \rho_0 Z^{k+1} - (1+k \rho_0) Z^k + 1 = 0, \quad \text{such that } |Z_0| > 1, \text{ but since } P(1) = 1.\]

It follows that,

\[C_0 = \frac{k}{1-(1/Z_0)}.
\]

Therefore,
\[
\mathbb{P}(Z) = \frac{(1 - Z^k)[1 - (1 / Z_0)]}{k (1 - Z)[1 - (Z / Z_0)]}.
\]

Thus,
\[
\mathbb{P}(Z) = (1 - Z^k) \left[ \frac{(1/k)}{1 - Z} + \frac{-1/(k Z_0)}{1 - (Z / Z_0)} \right].
\]

Denoting the inverse z-transform of the quantity \( \left[ \frac{(1/k)}{1 - Z} - \frac{1/(k Z_0)}{1 - (Z / Z_0)} \right] \) by \( f_n \), it follows that the inverse transform for \( \mathbb{P}(Z) \) must be:
\[
\mathbb{P}_n = f_n - f_{n+k} \quad (1.7)
\]

It is clear that,
\[
f_n = \begin{cases} 
(1/k) (1 - Z_0^{-n-1}) , & 0 \leq n \leq k - 1 \\
0 , & n < 0.
\end{cases} \quad (1.8)
\]

Recalling that \( Z_0 \) is a solution for the equation \( k \rho_0 Z^{k+1} - (1 + k \rho_0)Z^k + 1 = 0 \), it follows that,
\[
k \rho_0 (Z_0 - 1) = 1 - Z_0^{-k} \quad (1.9)
\]

By using equations (1.7), (1.8) and (1.9) yield:
\[
\mathbb{P}_n = \begin{cases} 
(1/k) (1 - Z_0^{-n-1}) , & 0 \leq n \leq k - 1 \\
\rho_0 (Z_0 - 1) Z_0^{-k-n-1} , & 0 \leq n \leq N.
\end{cases} \quad (1.10)
\]

Similarly, from the equations (1.3), (1.4), (1.5) and (1.6), we can get:
\[
\mathbb{P}_n = \rho_1 (Z_1 - 1) Z_1^{-k-n-1} , \quad \text{for } N < n \leq 2N,
\]

where \( \rho_1 = \lambda / 2 \mu \) and \( Z_1 \) is a solution for the equation:
\[
k \rho_1 Z^{k+1} - (1 + k \rho_1)Z^k + 1 = 0 .
\]

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and
\[ lP_n = \rho_j (Z_j - 1) Z_j^{k-n-1}, \quad \text{for } jN < n \leq (j+1)N, \quad (1.11) \]

where \( \rho_1 = \lambda / (j+1) \mu \) and \( Z_j \) is a solution for the equation:
\[ k \rho_j Z^{k-1} - (1 + k \rho_j) Z^k + 1 = 0, \]

it follows that,
\[ lP_n = \rho_{i-2} (Z_{i-2} - 1) Z_{i-2}^{k-n-1}, \quad \text{for } (i-2)N < n \leq (i-1)N, \quad (1.12) \]

where \( \rho_{i-2} = \lambda / (i-1) \mu \) and \( Z_{i-2} \) is a solution for the equation:
\[ k \rho_{i-2} Z^{k-1} - (1 + k \rho_{i-2}) Z^k + 1 = 0. \]

Finally,
\[ lP_n = \rho_{i-1} (Z_{i-1} - 1) Z_{i-1}^{k-n-1}, \quad \text{for } (i-1)N < n \leq K, \quad (1.13) \]

where \( \rho_{i-1} = \lambda / i \mu \) and \( Z_{i-1} \) is a solution for the equation:
\[ k \rho_{i-1} Z^{k-1} - (1 + k \rho_{i-1}) Z^k + 1 = 0. \]

By using equations (1.10), (1.11), (1.12) and (1.13), then the equation (1.1), will be as follows:
\[
P_m = \sum_{n=k-1}^{km-1} \frac{1}{k} (1 - Z_0^{n-1}), \quad \text{for } 0 \leq n \leq k-1, \]
\[
P_m = \sum_{n=k-1}^{km-1} \rho_0 (Z_0 - 1) Z_0^{k-n-1}, \quad \text{for } k-1 < n \leq N, \]
\[
\vdots \]
\[
P_m = \sum_{n=kN}^{kmN} \rho_j (Z_j - 1) Z_j^{k-n-1}, \quad \text{for } jN < n \leq (j+1)N, \quad (1.14) \]
\[
\vdots \]
\[
P_m = \sum_{n=(i-2)N}^{kmN} \rho_{i-2} (Z_{i-2} - 1) Z_{i-2}^{k-n-1}, \quad \text{for } (i-2)N < n \leq (i-1)N, \]
\[ P_m = \sum_{n=km}^{k(m+1)-1} \rho_m (Z_{l-1} - 1) Z_{l-1}^{k-n-1}, \text{ for } (i-1)N < n \leq K. \]

The expected number \( E(Q^*) \) of customers in the long run in the \( E_k/M/1 \) queueing system with only \((i-1)\) additional servers available to be added, is given by:

\[
E(Q^*) = \sum_{m=1}^{K} m \cdot P_m = \sum_{m=1}^{K} m \sum_{n=km}^{k(m+1)-1} \left[ \frac{1}{k} (1-Z_0)^{n-1} + \rho_0 (Z_0 - 1) Z_0^{k-n-1} + \cdots \right. \\
+ \rho_1 (Z_1 - 1) Z_1^{k-n-1} + \cdots + \rho_{i-1} (Z_{i-1} - 1) Z_{i-1}^{k-n-1} + \rho_i (Z_i - 1) Z_{i}^{k-n-1} \left. \right].
\]

Hence,

\[
E(Q^*) = \sum_{m=1}^{K} m \sum_{n=km}^{k(m+1)-1} \left[ \frac{1}{k} (1-Z_0)^{n-1} + \sum_{r=0}^{i-1} \rho_r (Z_r - 1) Z_r^{k-n-1} \right],
\]

where \( \rho_r = \lambda / (r+1) \mu ; \) \((r+1)\) is the number of servers in the system and \( Z_r \) is a solution for the equation \( k \rho_r Z^{k+1} - (1 + k \rho_r) Z^k + 1 = 0 ; \) \( r = 0, 1, \ldots, i-1. \) Using (1.14), it is evident that the probability that the number of customers in the system in the long run is greater than \( jN \) but less than or equal to \((j+1)N \) is given by:

\[
Pr[jN < Q^* \leq (j+1)N] = \sum_{n=km}^{k(m+1)-1} \rho_j (Z_j - 1) Z_j^{k-n-1} ; j = 0, 1, \ldots, (i-2).
\]

Similarly, it is evident that the probability that the number of customers in the system in the long run is greater than \((i-1)N \) is given by:

\[
Pr[Q^* > (i-1)N] = \sum_{n=km}^{k(m+1)-1} \rho_{i-1} (Z_{i-1} - 1) Z_{i-1}^{k-n-1}.
\]
It is clear that the probability that the $L^{th}$ additional server is busy is equal to the probability that the number of customers in the system is greater than $LN$, it is given by:

$$\Pr [Q^* > LN] = \sum_{r=L}^{k(m+1)-1} \sum_{n=kn} \rho_{r-1} (Z_{r-1} - 1) Z_{r-1}^{k-n-1} + \sum_{n=kn} \rho_{L-1} (Z_{L-1} - 1) Z_{L-1}^{k-n-1}. $$

Therefore,

$$\Pr (\text{the } L^{th} \text{ additional server is busy}) = \sum_{r=L}^{k(m+1)-1} \sum_{n=kn} \rho_{r-1} (Z_{r-1} - 1) Z_{r-1}^{k-n-1} + \sum_{n=kn} \rho_{L-1} (Z_{L-1} - 1) Z_{L-1}^{k-n-1}. $$

Let $W_n$ and $V_n$ be the waiting time until entrance to service and the service time of the $n^{th}$ customer, respectively. Denote the number of customers in the system left after the $n^{th}$ departure by $Q^*_n$. Using Little's equation which is given by:

$$\frac{1}{\lambda} E(Q^*) = E(W + V),$$

where $W = \lim_{n \to \infty} W_n$, $V = \lim_{n \to \infty} V_n$ and $Q^* = \lim_{n \to \infty} Q^*_n$.

it follows that the expected waiting time of a customer until leaving the system is given by:

$$E(W+V) = \frac{1}{\lambda} \left\{ \sum_{r=1}^{K} \sum_{m=1}^{k(m+1)-1} \frac{1}{k} (1 - Z_0)^{m-1} + \sum_{r=1}^{L-1} \rho_r (Z_r - 1) Z_r^{k-n-1} \right\}.$$ 

Let $C_1$ be the cost of a waiting customer and $C_2$ be the cost of an additional server. It is evident that $E(Q)$ the expected number of customers in the long run in the $E\mu/M/1$ queueing system and $\rho_0$ the traffic intensity, it is given by:
\[ \mathbb{E}(Q) = \sum_{n=1}^{K} \sum_{a=km}^{k(m+1)-1} \rho_{n}(Z_{n} - 1) Z_{n}^{k-n-1}. \]

It is obvious that whenever \( \mathbb{E}(Q) > \mathbb{E}(Q^*) \), where the quantity \( [\mathbb{E}(Q) - \mathbb{E}(Q^*)] \) is the expected number of customers waiting as a result of not permitting any additional server for a longer queue. It will be profitable to use additional servers, whenever their cost is less than that of the customers waiting due to not adding additional servers, in other words whenever,

\[
C_{1}\left\{ \sum_{n=1}^{K} \sum_{a=km}^{k(m+1)-1} \rho_{n}(Z_{n} - 1) Z_{n}^{k-n-1} - \sum_{n=km}^{K} \sum_{i=0}^{k(m+1)-1} \frac{1}{k} (1 - Z_{n})^{k-n-1} + \sum_{i=0}^{i-1} \rho_{i}(Z_{i} - 1) Z_{i}^{k-n-1} \right\}
\]

\[
> C_{2}\left\{ \sum_{j=1}^{i-1} \sum_{n=km}^{k(m+1)-1} \rho_{j+1}(Z_{j+1} - 1) Z_{j+1}^{k-n-1} + \sum_{n=km}^{k(m+1)-1} \rho_{j+1}(Z_{j+1} - 1) Z_{j+1}^{k-n-1} \right\}.
\]

Hence,

\[
C_{1}\left\{ \sum_{n=1}^{K} \sum_{a=km}^{k(m+1)-1} \left[ \rho_{n}(Z_{n} - 1) Z_{n}^{k-n-1} - \frac{1}{k} (1 - Z_{n})^{k-n-1} + \sum_{i=0}^{i-1} \rho_{i}(Z_{i} - 1) Z_{i}^{k-n-1} \right] \right\}
\]

\[
> C_{2}\left\{ \sum_{n=km}^{k(m+1)-1} \left[ \sum_{j=1}^{i-1} j \rho_{j+1}(Z_{j+1} - 1) Z_{j+1}^{k-n-1} + (i-1) \rho_{j+1}(Z_{j+1} - 1) Z_{j+1}^{k-n-1} \right] \right\}. \tag{1.15}
\]

Then it can be easily shown that \( [\mathbb{E}(Q) - \mathbb{E}(Q^*)] \) depends on the values of \( K, k, i, \rho_{i} \) and the values of \( Z_{n} \), where \( |Z_{n}| > 1 \); \( r = 0, 1, 2, \ldots, i-1 \). And it is obvious that the result of preceding method depends on the values of \( Z_{n} \), so we do not advise to use this method for large \( k \).

2. THE SECOND METHOD

From \([4]\), the probability that a customer departing after being served leaves behind \( n \) customers in the system \( \mathbb{E}/M/1 \) with finite waiting room is given by:

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\[ p_{(n)} = p_{\alpha k} + \ldots + p_{(n+1)k-1}, \quad n = 0, 1, \ldots, K, \quad (2.1) \]

as
\[ \sum_{m=0}^{(K+1)k-1} x^m p_m = \frac{\mu (1-x)}{\mu - x^k (\lambda - x \lambda + \mu)} \Delta(x) + x^{(K+1)k} R_2(x), \quad (2.2) \]

such that
\[ \lambda_n = \lambda_n \quad \text{for every } n \geq 0, \]

and
\[ \mu_n = (n+1) \mu \quad \text{for } jN < n \leq (j+1)N, \quad j = 0, 1, 2, \ldots, (i-1). \]

Substituting the appropriate values of \( \lambda_n \) and \( \mu_n \) in (2.2), it follows that:
\[ \sum_{m=0}^{(K+1)k-1} x^m p_m = \frac{\mu (1-x)}{\mu - x^k (\lambda - x \lambda + \mu)} \Delta(x) + x^{(K+1)k} R_2(x), \quad 0 \leq n \leq N \]
\[ \sum_{m=0}^{(K+1)k-1} x^m p_m = \frac{2\mu (1-x)}{2\mu - x^k (\lambda - x \lambda + 2\mu)} \Delta(x) + x^{(K+1)k} R_2(x), \quad N < n \leq 2N \]

\[ \sum_{m=0}^{(K+1)k-1} x^m p_m = \frac{j\mu (1-x)}{j\mu - x^k (\lambda - x \lambda + j\mu)} \Delta(x) + x^{(K+1)k} R_2(x), \quad (j-1)N < n \leq jN \]
\[ \sum_{m=0}^{(K+1)k-1} x^m p_m = \frac{(j+1)\mu (1-x)}{(j+1)\mu - x^k (\lambda - x \lambda + (j+1)\mu)} \Delta(x) + x^{(K+1)k} R_2(x), \quad jN < n \leq (j+1)N \]
\[
\sum_{m=0}^{(K+1)k-1} x^m p_m = \frac{(i-1) \mu (1-x)}{(i-1) \mu - x^k (\lambda - x \lambda + (i-1) \mu)} \frac{\Delta(x)}{\Delta} + x^{(K+1)k} R_2(x),
\]
\[\text{for } (i-2)N < n \leq (i-1)N.\]

\[
\sum_{m=0}^{(K+1)k-1} x^m p_m = \frac{i \mu (1-x)}{i \mu - x^k (\lambda - x \lambda + i \mu)} \frac{\Delta(x)}{\Delta} + x^{(K+1)k} R_2(x),
\]
\[\text{for } (i-1)N < n \leq K.\]

where \(p_m\) is the coefficient of \(x^m\), obtained from the first term only in the right hand side of the previous last equations. Replacing the values of \(p_m\) in the right hand side of equation (2.1), it follows that:

\[
p_{[n],1} = p_{hk,1} + \ldots + p_{(n+1)k-1,1}, \quad 0 \leq n \leq N
\]

\[
p_{[n],2} = p_{hk,2} + \ldots + p_{(n+1)k-1,2}, \quad N < n \leq 2N
\]

\[
\vdots
\]

\[
p_{[n],j} = p_{hk,j} + \ldots + p_{(n+1)k-1,j}, \quad (j-1)N < n \leq jN
\]

\[
p_{[n,j+1]} = p_{hk,(j+1)} + \ldots + p_{(n+1)k-1,(j+1)}, \quad jN < n \leq (j+1)N
\]

\[
\vdots
\]

\[
p_{[n],(i-1)} = p_{hk,(i-1)} + \ldots + p_{(n+1)k-1,(i-1)}, \quad (i-2)N < n \leq (i-1)N
\]

\[
p_{[n],i} = p_{hk,i} + \ldots + p_{(n+1)k-1,i}, \quad (i-1)N < n \leq K
\]

The expected number \(E(Q^*)\) of customers in the long run in the \(E_k/M/1\) queueing system with only \((i-1)\) additional servers available to be added is given by:

\[
E(Q^*) = \sum_{n=1}^{K} np_{[n]},
\]

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then,

\[ E(Q^*) = \sum_{n=1}^{N} np_{n,1} + \sum_{n=N+1}^{2N} np_{n,2} + \cdots \]

\[ + \sum_{n=(j-1)N+1}^{jN} np_{n,j} + \sum_{n=jN+1}^{(j+1)N} np_{n,j+1} + \cdots + \sum_{n=(i-2)N+1}^{(i-1)N} np_{n,(i-1)} + \sum_{n=(i-1)N+1}^{K} np_{n,i} \]

Hence,

\[ E(Q^*) = \sum_{\sigma=1}^{i} \sum_{n=(\sigma-1)N+1}^{\sigma N} \sum_{r=0}^{n} \sum_{t=0}^{(n+1)k-1} p_{r,t} \quad (2.4) \]

By using equation (2.3), it is evident that the probability that the number of customers in the system in the long run is greater than \( jN \) but less than or equal to \( (j+1)N \) is given by:

\[ \Pr [ jN < Q^* \leq (j+1)N ] = \frac{(j+1)^N}{n=jN+1} \sum_{n=jN+1}^{(j+1)N} \sum_{r=nk}^{(n+1)k-1} p_{r,t} \quad (2.5) \]

where \( j = 0, 1, 2, \ldots, (i-2) \).

Similarly, it is evident that the probability that the number of customers in the system in the long run is greater than \( (i-1)N \) and less than or equal to \( K \) is given by:

\[ \Pr [ (i-1)N < Q^* \leq K ] = \frac{K}{n=(i-1)N+1} \sum_{n=iN+1}^{(i+1)N} \sum_{c=ck}^{(c+1)k-1} p_{r,d} \quad (2.6) \]

It is clear that the probability that the \( L^{th} \) additional server is busy is equal to the probability that the number of customers in the system is greater than \( LN \) and less than or equal to \( K \), it is given by:
\[
\Pr (LN < Q^* \leq K) = \sum_{r=1}^{i-2} \sum_{n=N+1}^{(r+1)N} p_{n[r+1]} + \sum_{n=(i-1)N+1}^{K} p_{n[i]}. 
\]

Therefore,

\[
\Pr \text{ (the } L^{th} \text{ additional server is busy)} = \sum_{r=1}^{i-2} \sum_{n=N+1}^{(r+1)N} \sum_{n=N}^{(n+1)k-1} p_{n[r+1]} + \sum_{n=(i-1)N+1}^{K} \sum_{r=N}^{(n+1)k-1} p_{n[r]}.
\]

It is clear that,

\[
\Pr \text{ (the } (i-1)^{th} \text{ additional server is busy)} = \sum_{n=(i-1)N+1}^{K} \sum_{r=N}^{(n+1)k-1} p_{n[r]}.
\]

The expected waiting time of a customer until leaving the system is given by:

\[
E(W+V) = \frac{1}{\lambda} E(Q^*),
\]

then,

\[
E(W+V) = \frac{1}{\lambda} \sum_{r=1}^{i-2} \sum_{n=(r+1)N+1}^{N} n \sum_{r=N}^{(n+1)k-1} p_{r,r}.
\]

It is evident that,

\[
E(Q) = \sum_{n=1}^{K} n p_{n} = \sum_{n=1}^{K} n \sum_{r=N}^{(n+1)k-1} p_{r}, \tag{2.7}
\]

where

\[
\lambda_n = \lambda, \quad \mu_n = \mu \quad \text{for every } n \geq 0, \quad \text{and} \quad \rho = \lambda / \mu.
\]

Consequently,
\[ C_1 \{ E(Q) - E(Q^*) \} > C_2 \left\{ \sum_{j=1}^{i-2} j \Pr[jN < Q^* \leq (j+1)N] \right\} \\
+ (i-1) \Pr[(i-1)N < Q^* \leq K]. \]  

(2.6)

Substituting from the equations (2.4), (2.5), (2.6), and (2.7) into (2.8), then the inequality will become as follow:

\[ C_1 \left\{ \sum_{n=1}^{K} \sum_{r=nk}^{(n+1)k-1} \sum_{m=1}^{i} \sum_{n=(i-1)N+1}^{N} \sum_{s=mk}^{(m+1)k-1} P_{r,s} \right\} \\
> C_2 \left\{ \sum_{j=1}^{i-2} j \sum_{n=jk+1}^{N} \sum_{r=sk}^{(s+1)k-1} P_{r,s} \right\} \\
+ (i-1) \sum_{n=(i-1)N+1}^{K} \sum_{s=sk}^{(s+1)k-1} P_{r,s}. \]  

(2.9)

It can be easily shown that \( \{ E(Q) - E(Q^*) \} \) depends on the values of \( K, k, i, N \) and the values of the probabilities \( p_r, p_{r,s} \).

CONCLUSION

Finally, if we compare both results of the two methods in (1.15) and (2.9), we can see clearly that the main factor which affects our results is the cost (both \( C_1, C_2 \)). Furthermore, the most important factors that affect the preference between \( E_x/M/1 \) queueing system with additional servers and without additional server, in both methods, are the same; we can name them to be: The waiting room size, the arrival stages and the maximum number of servers allowed in the system. Also, it is obvious that the remaining different factors in our results are due to the different in solving algorithms.
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مجلة البحوث الهندسية
ببورسعيد

تصدرها كلية الهندسة ببورسعيد
جامعة قناة السويس

المجلد الثاني - العدد الثاني
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